Efficient Vibration Mode Analysis of Aircraft with Multiple External Store Configurations

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A coupling method for efficient vibration mode analysis of aircraft with multiple external store configurations is presented. A set of low-frequency vibration modes, including rigid-body modes, represent the aircraft. Each external store is represented by its vibration modes with clamped boundary conditions, and by its rigid-body inertial properties. The aircraft modes are obtained from a finite-element model loaded by dummy rigid external stores with fictitious masses. The coupling procedure unloads the dummy stores and loads the actual stores instead. The analytical development is presented, the effects of the fictitious mass magnitudes are discussed, and a numerical example is given for a combat aircraft with external wing stores. Comparison with vibration modes obtained by a direct (full-size) eigensolution shows very accurate coupling results. Once the aircraft and stores data bases are constructed, the computer time for analyzing any external store configuration is two to three orders of magnitude less than that of a direct solution.

Nomenclature

$\{F\}$	= force vector
$[I]$	= unit matrix
$[K]$	= stiffness matrix
$[m]$	= generalized mass matrix
$[M]$	= mass matrix
$\{u\}$	= discrete displacement vector of the combined structure
$\{x\}$	= discrete displacement vector of the aircraft before coupling with external stores
$[\Psi]$	= eigenvector matrix of the uncoupled aircraft and stores
$\{\xi\}$	= displacement vector in generalized coordinates
ω	= natural frequency

Subscripts and Superscripts

A	= aircraft (substructure A)
B	= store (substructure B)
F	= fictitious masses
I	= interface coordinates
R	= rigid-body modes
$[\ ]$	= diagonal matrix
$[\ ]^T$	= transposed matrix

I. Introduction

A FIGHTER aircraft is a complex structural system that is required to carry various configurations of external stores (such as fuel tanks and military stores). With several store stations, and a variety of external stores that can be carried on each station, there are thousands of combinations of possible external store loadings. Each store configuration must be evaluated for flutter and cleared to safe flight speeds. Therefore, it is necessary to use efficient computational tools with which the flutter evaluation can be completed within a reasonable cost and time duration.

The scope of this paper is vibration modes analysis. The fact that the basic aircraft structure does not change with each store configuration calls for using the modal coupling method. With this method, a limited set of low-frequency vibration modes of the basic aircraft would be coupled with vibration modes of any combination of external stores. The low-order coupling equation would then be solved for the vibration modes of the entire system.

The first method of partial modal coupling was presented by Hunn,[1] and a matrix formulation of this method was given by Gladwell.[2] The method was developed for statically determinate beam-type models. Hurty[3-5] is credited with the development of a coupling method applicable to redundant complex structures. Hurty's method is based on the assumption that the motion of a substructure is a linear combination of three types of modes:

1) Fixed constraint modes, which are determined for each substructure by fixing all of its interface coordinates.

2) Redundant constraint modes, which are determined by applying a unit displacement at each interface coordinate while all others are fixed, and then calculating the resulting displacements of the substructure.

3) Rigid-body modes.

A different approach was taken by Goldman[6] and by Hou.[7] The substructure modes are calculated with free interface coordinates. This method avoids the use of various types of substructure modes. However, the number of modes needed to achieve a given accuracy is higher and numerical results are often incorrect. Benfield and Hruda[8] modified the coupling method by using free interface coordinates subjected to external loading. The interface coordinates of a substructure are loaded by the reduced stiffness and mass matrices of the neighboring substructure. This approach is a good compromise between the fully constrained and the free-free interface methods, but the reduction and the coupling procedures are not efficient. Karpel and Newman[9] introduced fictitious masses to the interface coordinates of the substructures. This approach increased the coupling accuracy while using free-free modes only. Meirovitch and Hale[10,11] introduced a different approach in which the motion of each substructure is represented by admissible functions.

The coupling method of this paper develops the fictitious mass idea of Ref. 9 and applies it to aircraft-store coupling. The problem definition, basic assumptions, analytical development, and application to aircraft with multiple external store configurations are given in Sec. II. A discussion on the numerical results of Ref. 9 and a new numerical example of a fighter aircraft with external stores are given in Sec. III. Conclusions and guidelines for proper use of this method are given in Sec. IV.

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II. Analytical Development

A. Definition of the Problem

The structural system consists of a central structure and additional substructures. The connection of the substructures to the central structure is statically determinate. For convenience, the terminology of "basic aircraft" for the central structure and "external stores" for the substructures will be used in the rest of the paper. A scheme of an aircraft with two external stores is given in Fig. 1.

It is assumed that a finite-element model of the aircraft is given, that the connections to the stores are defined, and that the external store mass distributions are known. The natural frequencies and the associated mode shapes of the stores with clamped boundary conditions are known from analysis or a ground vibration test.

The problem is to calculate the dynamic properties (a set of low frequencies, vibration modes, and generalized masses) of the aircraft with any combination of external stores, with sufficient accuracy and with minimum cost and computer time.

B. Basic Assumptions

The following basic assumptions are made:

1) The structures are undamped and linear.

2) The discrete displacements of the aircraft are a linear combination of a limited set of low-frequency modes of the basic uncoupled aircraft (including rigid-body modes). The validity of this assumption is strongly dependent on the type and number of the aircraft vibration modes that are taken into account. The external stores impose local distortions of the aircraft at the vicinity of the connection points. The basic aircraft modes should include such type of distortions. This is obtained by using a set of fictitious masses loaded at the connections to the store degrees of freedom.

3) The discrete displacements of each external store are a linear combination of a limited set of low-frequency modes of the store with clamped boundary conditions, and of its free-free rigid-body modes.

4) The connections of an external store to the aircraft are statically determinate, which imposes the number of the connection coordinates to be equal to the number of the store free-free rigid-body modes. This assumption is valid for dynamics analysis of most types of external stores.

C. The Coupling Equations

The basic aircraft (denoted by A), loaded with fictitious masses $[M_F]$ at the store connection degrees of freedom (denoted by I), is first considered. The free matrix equation of motion may be written in the form:

$$\begin{bmatrix} M_{AA} & M_{AI} \\ M_{AI}^T & M_{II}^{(A)} + M_F \end{bmatrix} \begin{Bmatrix} \ddot{x}_A \\ \ddot{x}_I \end{Bmatrix} + \begin{bmatrix} K_{AA} & K_{AI} \\ K_{AI}^T & K_{II}^{(A)} \end{bmatrix} \begin{Bmatrix} x_A \\ x_I \end{Bmatrix} = \{0\} \tag{1}$$

An eigenvalue routine can now be used to calculate a set of lowest frequencies and the corresponding eigenvectors and generalized masses associated with Eq. (1), where:

$$[\dot{m}_A] = [\Psi_A^T \; \Psi_I^T] \begin{bmatrix} M_{AA} & M_{AI} \\ M_{AI}^T & M_{II}^{(A)} + M_F \end{bmatrix} \begin{bmatrix} \Psi_A \\ \Psi_I \end{bmatrix} \tag{2}$$

$$[\dot{\omega}_A^2 m_A] = [\Psi_A^T \; \Psi_I^T] \begin{bmatrix} K_{AA} & K_{AI} \\ K_{AI}^T & K_{II}^{(A)} \end{bmatrix} \begin{bmatrix} \Psi_A \\ \Psi_I \end{bmatrix} \tag{3}$$

The free equation of motion of the external store, connected to the gound (clamped) instead of to the aircraft, is now considered:

$$[M_{BB}]\{\ddot{x}\} + [K_{BB}]\{x\} = \{0\} \tag{4}$$

An eigenvalue routine is used to calculate a set of lowest frequencies and the corresponding eigenvectors and generalized masses, associated with Eq. (4), where:

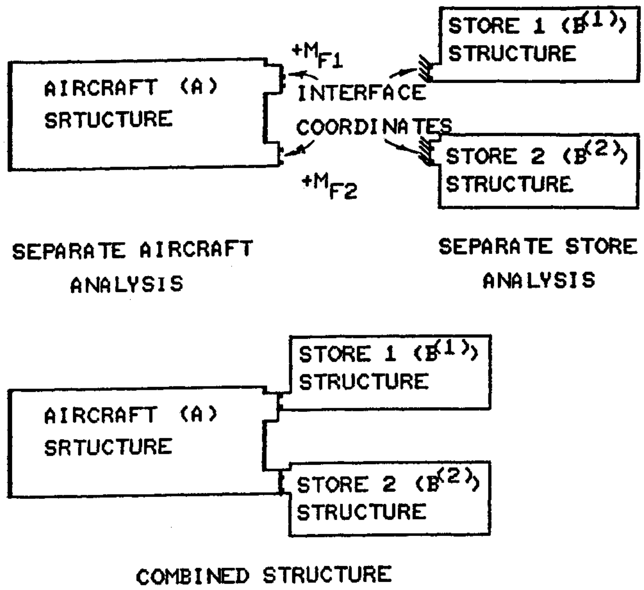

Fig. 1 A scheme of an aircraft with two external stores.

$$[\dot{m}_B] = [\Psi_B]^T[M_{BB}][\Psi_B] \tag{5}$$

$$[\dot{\omega}_B^2 m_B] = [\Psi_B]^T[K_{BB}][\Psi_B] \tag{6}$$

The geometric properties of the store are then used to obtain a complete set of free-free rigid-body modes $[\Psi_{RB}]$. Since the connections to the aircraft are statically determinate, the number of the rigid-body modes is equal to the number of connection degrees of freedom. The rigid-body modes of the store may be defined such that the portion of $[\Psi_{RB}]$ associated with the interface coordinates is

$$[\Psi_{RIB}] = [I] \tag{7}$$

The combined system of the basic aircraft plus an actual store is now considered. The equation of motion for the basic aircraft (without fictitious masses) may be written in the form:

$$\begin{bmatrix} M_{AA} & M_{AI} \\ M_{AI}^T & M_{II}^{(A)} \end{bmatrix} \begin{Bmatrix} \ddot{u}_A \\ \ddot{u}_I \end{Bmatrix} + \begin{bmatrix} K_{AA} & K_{AI} \\ K_{AI}^T & K_{II}^{(A)} \end{bmatrix} \begin{Bmatrix} u_A \\ u_I \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_I \end{Bmatrix} \tag{8}$$

where $\{F_I\}$ is the vector of forces applied by the store on the aircraft at the interface coordinates. The displacements of Eq. (8) are assumed to be a linear combination of the eigenvectors obtained from Eq. (1):

$$\begin{Bmatrix} u_A \\ u_I \end{Bmatrix} = \begin{bmatrix} \Psi_A \\ \Psi_I \end{bmatrix} \{\xi_A\} \tag{9}$$

Substitution of Eq. (9) into Eq. (8), and premultiplication by $[\Psi_A^T \; \Psi_I^T]$[while considering Eqs. (2) and (3)] gives:

$$\Big[[\dot{m}_A] - [\Psi_I]^T[M_F][\Psi_I]\Big]\{\ddot{\xi}_A\} + [\dot{\omega}_A^2 m_A]\{\xi_A\} = [\Psi_I]\ \{F_I\} \tag{10}$$

The equation of motion for a store, as part of the combined aircraft-store system, is now considered. The forces applied on the stores are equal in magnitude and opposite in direction to those applied on the aircraft in Eq. (8). Thus, the equation of motion for a store (as part of the combined system) is

$$\begin{bmatrix} M_{II}^{(B)} & M_{IB} \\ M_{IB}^T & M_{BB} \end{bmatrix} \begin{Bmatrix} \ddot{u}_I \\ \ddot{u}_B \end{Bmatrix} + \begin{bmatrix} K_{II}^{(B)} & K_{IB} \\ K_{IB}^T & K_{BB} \end{bmatrix} \begin{Bmatrix} u_I \\ u_B \end{Bmatrix} = \begin{Bmatrix} -F_I \\ 0 \end{Bmatrix} \tag{11}$$

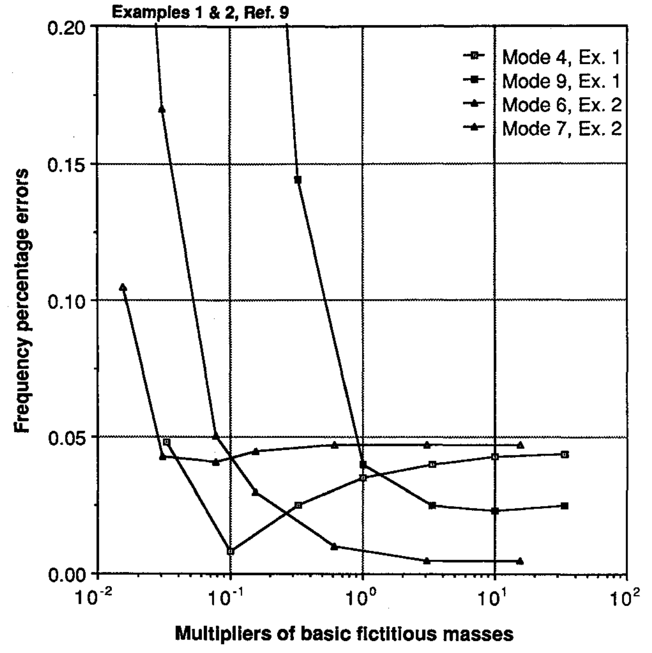

Fig. 2 Frequency percentage errors vs multipliers of fictitious masses.

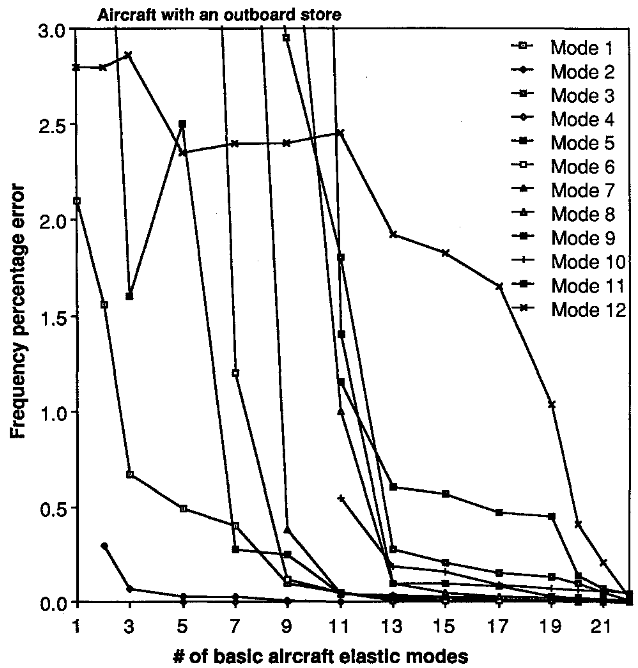

Fig. 3 Aircraft-store coupling frequency percentage errors vs number of elastic modes.

The displacements of Eq. (11) are assumed to be a linear combination of the clamped and rigid-body modes of the store, which implies

$$\begin{Bmatrix} u_I \\ u_B \end{Bmatrix} = \begin{bmatrix} I & 0 \\ \Psi_{RB} & \Psi_B \end{bmatrix} \begin{Bmatrix} \xi_{RB} \\ \xi_B \end{Bmatrix} \tag{12}$$

Eqs. (5), (6), (11), and (12)

$$\begin{bmatrix} m_R & m_{RB} \\ m_{RB}^T & \grave{m}_B \end{bmatrix} \begin{Bmatrix} \ddot{\xi}_{RB} \\ \ddot{\xi}_B \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \grave{\omega}_B^2 m_B \end{bmatrix} \begin{Bmatrix} \xi_{RB} \\ \xi_B \end{Bmatrix} = \begin{Bmatrix} -F_I \\ 0 \end{Bmatrix} \tag{13}$$

where

$$[m_{RB}] = \Big[[M_{IB}] + [\Psi_{RB}]^T [M_{BB}] \Big] [\Psi_B] \tag{14}$$

$$[m_R] = [I \ \Psi_{RB}^T] \begin{bmatrix} M_{II}^{(B)} & M_{IB} \\ M_{IB}^T & M_{BB} \end{bmatrix} \begin{bmatrix} I \\ \Psi_{RB} \end{bmatrix} \tag{15}$$

It should be noted that, generally, the rigid-body mass matrix $[m_R]$ is not diagonal. The zero terms in the stiffness matrix of Eq. (13) are due to no elastic distortion in rigid-body modes.

It is clear that the interconnection displacements of Eq. (9) are equal to those of Eq. (12), which implies

$$\{\xi_{RB}\} = [\Psi_I]\{\xi_A\} \tag{16}$$

Substitution of Eq. (16) into Eq. (13) and premultiplication of the upper terms of Eq. (13) by $[\Psi_I]^T$ gives:

$$\begin{bmatrix} [\Psi_I]^T [m_R][\Psi_I] & [\Psi_I]^T [m_{RB}] \\ [m_{RB}]^T [\Psi_I] & [\grave{m}_B] \end{bmatrix} \begin{Bmatrix} \ddot{\xi}_A \\ \ddot{\xi}_B \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \grave{\omega}_B^2 m_B \end{bmatrix} \begin{Bmatrix} \xi_A \\ \xi_B \end{Bmatrix} = \begin{Bmatrix} -[\Psi_I]^T\{F_I\} \\ 0 \end{Bmatrix} \tag{17}$$

Substitution of Eq. (10) into Eq. (17) gives the homogeneous coupling equation of motion with one external store

$$\begin{bmatrix} [\grave{m}_A] + [\Psi_I]^T([m_R] - [M_F])[\Psi_I] & [\Psi_I]^T[m_{RB}] \\ [m_{RB}]^T[\Psi_I] & [\grave{m}_B] \end{bmatrix} \begin{Bmatrix} \ddot{\xi}_A \\ \ddot{\xi}_B \end{Bmatrix} + \begin{bmatrix} \grave{\omega}_A^2 m_A & 0 \\ 0 & \grave{\omega}_B^2 m_B \end{bmatrix} \begin{Bmatrix} \xi_A \\ \xi_B \end{Bmatrix} = \{0\} \tag{18}$$

Equation (18) can be used for more than one store by expanding the store related terms

$$\{\xi_B\} = \begin{Bmatrix} \xi_B^{(1)} \\ \xi_B^{(2)} \\ \vdots \end{Bmatrix} \tag{19}$$

$$[\grave{m}_B] = \begin{bmatrix} [\grave{m}_B]^{(1)} & & 0 \\ & [\grave{m}_B]^{(2)} & \\ 0 & & \ddots \end{bmatrix} \tag{20}$$

$$[\grave{\omega}_B^2 m_B] = \begin{bmatrix} [\grave{\omega}_A^2 m_B]^{(1)} & & 0 \\ & [\grave{\omega}_A^2 m_B]^{(2)} & \\ 0 & & \ddots \end{bmatrix} \tag{21}$$

$$[\Psi_I]^T[m_{RB}] = \big[[\Psi_I]^T[m_{RB}]^{(1)} \ [\Psi_I]^T[m_{RB}]^{(2)} \ldots \big] \tag{22}$$

$$[\Psi_I]^T([m_R] - [M_F])[\Psi_I] = \sum_j \big[[\Psi_I]^T([m_R] - [M_F])[\Psi_I] \big]^{(j)} \tag{23}$$

The basic aircraft vibration modes (with fictitious masses) and the store vibration modes (with clamped boundary conditions) serve as generalized coordinates in Eq. (18). Some of the basic aircraft modes are rigid-body modes. Until now, these modes were treated as part of the aircraft vibration modes. The only difference is that these modes have zero frequency that causes singularity in the stiffness matrix (the corresponding values in $[\grave{\omega}^2 m_A]$ are zero). A vibration mode analysis routine, which takes into account the presence of rigid-body modes, can now be used to find the coupled natural frequencies, vibration modes in generalized coordinates, and generalized masses of Eq. (18). Equation (16) is then used to calculate the store rigid-body participation in the coupled modes. Once the coupled modes in generalized coordinates are defined, one can calculate the coupled modes in discrete coordinates using the transformations of Eqs. (9) and (12).

It is also possible to use the coupled modes (in generalized coordinates) of Eq. (18) for flutter analysis by coupling. This feature results in additionally significant computer time savings in repetitive store flutter analysis.

III. Numerical Examples

A. Effect of Fictitious Mass Magnitudes

The effect of fictitious mass magnitudes can be learned from the numerical examples of Ref. 9. Even though the method of Ref. 9 is applicable to arbitrary substructure interconnection, its first two examples are with statically determinate interconnections. The method of Ref. 9 also differs from the method of this paper in the way substructure B is treated. However, there is no difference in the way substructure A (to which fictitious masses are added) is represented in the coupling equations. More details about these examples are given in Ref. 12.

Some details taken from Refs. 9 and 12 are repeated here for clarity and completeness. Also, key results are shown here in a different way that is consistent with the notations and arguments of this paper. A frequency percentage error, for comparing the coupling results with direct (full size) solution, is defined by

$$E_{\omega_i} = \frac{\omega_{i_{\text{coupling}}} - \omega_{i_{\text{direct}}}}{\omega_{i_{\text{direct}}}} \times 100 \tag{24}$$

The first example of Ref. 9 is a one-dimensional mass-spring system, divided into two substructures with 11 mass-loaded degrees of freedom each, including one interface coordinate. Seven lowest frequency modes of substructure A are coupled with the modes of substructure B to produce 11 lowest frequency modes of the entire structure, including one rigid-body mode. The weight of each substructure is about 30 kg. In the present paper notation, that would be the value of $[m_R]$ of Eq. (15). Having this value as a basic fictitious mass, variations of the fourth and the ninth elastic mode frequency percentage errors vs multipliers of the basic fictitious mass are shown in Fig. 2. These modes have the highest errors of this example.

The second example of Ref. 9 is a two-dimensional straight beam, divided into two substructures with eight grid points each. Each grid point has two degrees of freedom, vertical translation, and in-plane rotation. The rigid-body mass matrix ($[m_R]$) of substructure B has diagonal terms of 2700 kg and 47,000 kg-m^2 and off-diagonal terms of 8000 kg-m. The eight lowest frequency modes (out of 16) of substructure A are coupled with the modes of substructure B to produce the nine lowest frequency modes of the entire structure, including two rigid-body modes. A basic fictitious mass matrix $[M_F]$ is defined here with diagonal terms of 3250 kg and 32,500 kg-m^2 and zero off-diagonal terms. The variations of the sixth and seventh elastic mode frequency percentage errors vs scalar multipliers of the basic fictitious mass matrix are also shown in Fig. 2. Again, these modes have the highest errors of this example.

It can be learned from Fig. 2 that, with multipliers of more than 1.0, the coupling results are very accurate (frequency error of less than 0.05%) and insensitive to the multiplier value. One should take into account, however, that very high multipliers may cause numerical problems in the eigensolution due to matrix ill-conditioning. On the other hand, with multipliers of less than 1.0, the accuracy may deteriorate rapidly. The second example also indicates that the ratios between different rigid-body mass terms of substructure B do not have to be preserved in $[M_F]$, and that off-diagonal terms may be neglected. These indications will be tested in the following aircraft-store numerical example.

B. Aircraft with External Stores

The numerical example is of a fighter aircraft with two wing store stations. The detailed NASTRAN stiffness model (half aircraft with symmetric boundary conditions) yields thousands degrees of freedom. Guyan reduction was used to reduce the problem size to several hundred degrees of freedom, retaining the mass-loaded degrees of freedom and the store connections. This reduced model was used to calculate the basic aircraft symmetric modes (with dummy stores). This model was also used for direct analysis of aircraft with actual external stores, for comparison with the coupling results.

The basic aircraft modes were coupled with clamped modes and rigid-body modes of an external store to yield 15 vibration modes (up to about 50 Hz), including three rigid-body modes, of an aircraft with one external store (on each wing) at the outboard station and no store at the inboard station. In this case, the procedure removes the fictitious masses from the inboard station without adding an actual store instead.

The outboard store is connected to the wing at two grid points. The forward point has two connection coordinates: Y (spanwise) and Z (upward). The rear point has four connection coordinates: X (chordwise), Y, Z, and θ_x. The basic fictitious mass matrix $[M_F]$ has been chosen to be diagonal. The ratios between the terms of $[M_F]$ and the respective diagonal terms of the rigid-body mass matrix $[m_R]$ of the actual outboard store are 4.7, 2.9, 5.7, 6.0, 4.3, and 0.83. The inboard and outboard stations were loaded by the same fictitious mass matrices.

A study of the variation of coupled natural frequencies and generalized masses with the number of basic aircraft low-frequency elastic modes taken into account has been conducted. Three aircraft rigid-body modes, six store rigid-body modes and three store elastic modes were taken into account in addition to the aircraft elastic modes.

A generalized mass percentage error (where modes are normalized to unit maximum displacements) is defined by

$$E_{m_i} = \frac{m_{i_{\text{coupling}}} - m_{i_{\text{direct}}}}{m_{i_{\text{direct}}}} \times 100 \tag{25}$$

Variations of frequency percentage errors with the number of basic aircraft elastic modes taken into account (n) are given in Fig. 3. The frequency and generalized mass percentage errors with 19 and 22 basic aircraft elastic modes are given in Table 1. The frequency errors with $n = 19$ and with the basic fictitious mass matrix multiplied by 10.0 are also given in Table 1.

It is clear that with 22 basic aircraft elastic modes (up to about 100 Hz), the results are very accurate for the purpose of low-frequency structural dynamics analysis such as flutter. It can also be observed that the errors converge slower for the higher coupled frequencies, and that the multiplication of the basic fictitious mass matrix by 10.0 improves the coupling results. The generalized mass errors of modes 11 and 12 in the 19 basic aircraft modes case of Table 1 are large, similar in magnitude, and opposite in sign. This is typical to close-frequency modes with strong coupling between different structural components. With 22 basic aircraft modes these errors are reduced to an acceptable level.

Based on this case, and many other analyzed cases, it is recommended that the number of uncoupled elastic modes taken into account will be at least twice the number of the required coupled elastic modes, and that the frequency range of the uncoupled modes will be at least twice the coupled frequency range of interest. It is also recommended that the values of the fictitious masses loading a certain station will be at least those of the diagonal terms in $[m_R]$ of the heaviest store loading this station.

Table 1 Aircraft-store coupling frequency and generalized mass percentage errors

	E_{ω_i}			E_{m_i}	
n:	19	22	19	19	22
$[M_F]$ multiplier:	1.0	1.0	10.0	1.0	1.0
Mode					
1	0.01	0.00	0.00	−0.77	0.00
2	0.00	0.00	0.00	0.03	0.00
3	0.13	0.01	0.02	−1.89	−0.15
4	0.00	0.00	0.00	0.05	0.00
5	0.01	−0.01	0.00	0.24	0.12
6	0.01	0.00	0.00	−0.09	−0.14
7	0.02	0.01	0.02	2.67	−0.25
8	0.02	0.01	0.06	0.72	0.00
9	0.03	0.01	0.07	−0.52	−0.16
10	0.07	0.05	0.10	5.53	0.92
11	0.45	0.04	0.09	49.86	−3.32
12	1.03	0.02	0.11	−48.17	4.11

The CPU time for calculating the direct NASTRAN solution (15 modes) was about 150 s (on a CDC mainframe). The CPU time for calculating 25 basic aircraft and store modes by NASTRAN (to create the coupling input) was about 200 s. However, the coupling run took less than 1 s. It is obvious that the CPU time and turn-around time savings for analyzing many store configurations or performing sensitivity studies, with an unchanged basic aircraft structure and fictitious masses, are very significant.

IV. Conclusions

1) The coupling procedure for vibration modes of aircraft with external stores is very accurate for dynamic analyses which involve low-frequency modes of the coupled structure (such as flutter).

2) The procedure is very efficient for analyzing many external store configurations (installed on the same basic aircraft) using the same set of fictitious masses for all cases.

3) The fictitious masses which load the store attachment degrees of freedom should be similar or larger than those of the largest possible store mounted on this store station. With large fictitious masses, the procedure is insensitive to their exact values, providing they are not large enough to cause ill-conditioning.

4) The frequency range of the basic aircraft modes, which serves as input for the coupling procedure, should be higher than the output frequency range of interest. An input/output frequency range factor of 2 or more is recommended.

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